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SURFACE ACOUSTIC WAVES NEAR BODIES COATED WITH AN ABSORBING LAYER†

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Waves which occur on the surface of solids coated with a thin compressible layer and which attenuate rapidly with distance from the surface are considered. The absorbing layer is modelled by a non-classical boundary condition [1]. It is shown that waves can propagate near the body and are localized inside a certain layer whose thickness depends, in particular, on the mechanical properties of the coating.

1. SURFACE WAVES ON A PLANE

SUPPOSE the $y=0$ plane is coated with a thin absorbing layer, while the space $y>0$ is filled with an ideal compressible liquid. We will consider acoustic waves propagating near the absorbing layer.

Outside the layer the pressure in the acoustic wave is described by the following equation

$$\partial^2 p/\partial t^2 = c_0^2(\partial^2 p/\partial x^2 + \partial^2 p/\partial y^2) \tag{1.1}$$

The boundary condition on the solid, which models the absorbing coating, has the form [1–4]

$$y=0,\quad \partial^2 p/\partial t^2 - G\partial p/\partial y = c_*^2\partial^2 p/\partial x^2,\quad G=\rho_* c_*^2/(h_0\rho_0) \tag{1.2}$$

where c_* and c_0 are the velocity of sound in the absorbing layer and outside it, respectively, ρ_*, ρ_0 are the mean densities of the liquid in the absorbing layer and outside it, and h_0 is the mean thickness of the layer. If $c_* \ll c_0$, the right-hand side in condition (1.2) must be equated to zero [1].

The constant G represents the compressibility of the absorbing layer. If it is identified with the acceleration due to gravity, then, when $c_* \ll c_0$ condition (1.2) is identical with the boundary condition on the free surface of the liquid [2, 5]. It is natural to expect that the solutions of (1.1) and (1.2) will be similar to gravitational waves on the surface of the liquid. Hence, we will seek solutions which decay exponentially with distance from the surface, i.e. waves of the form

$$\begin{aligned} &p=\exp[-i\omega t + i(k_x x + k_y y)],\quad k_x^2+k_y^2=\omega^2/c_0^2 \\ &k_x>\omega/c_0,\quad k_y^2=\omega^2/c_0^2-k_x^2=(i\gamma)^2=-\gamma^2,\quad \gamma\geqslant 0 \end{aligned} \tag{1.3}$$

Everywhere henceforth the time dependence is chosen in the form $\exp(-i\omega t)$.

Boundary condition (1.2) takes the form

$$y=0,\quad -\omega^2+G\gamma=-c_*^2k_x^2 \tag{1.4}$$

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and defines the law of dispersion of the surface waves. It can be seen that for each frequency ω there is a surface wave of the form (1.3) $(ik_y = -\gamma)$, which propagates along the wall with parameters k_x and γ, defined by the equations

$$k_x^2 = (2\omega^2 c_*^2 - G\sqrt{4\omega^2 c_*^2 c + G^2} + G^2)/(2c_*^4)$$
$$\omega^2 = (\gamma^2 c_*^2 + G\gamma)/c, \quad c = 1 - c_*^2/c_0^2 \tag{1.5}$$

When $c_* \ll c_0$ we have $\omega^2 = G\gamma$.

The parameter γ defines the effective thickness of the layer l in which the surface wave is localized $l = 1/\gamma$. This wave transfers energy only along the absorbing layer.

The energy flux density can be calculated from the formula

$$q = (p^* \upsilon + p\upsilon^*)/2 \tag{1.6}$$

using the linearized Euler equation

$$\partial\upsilon/\partial t = -\rho_0^{-1}\,\text{grad}\,p$$

For surface waves of the form (1.3) with amplitude A, we obtain

$$q = (A^2 k_x (\rho_0\omega)^{-1} \exp(-2\gamma y), 0)$$

The total energy flux through the surface $x = \text{const}$, transferred by such a wave, is

$$Q = \int_0^{+\infty} q_x dy = \frac{A^2 k_x}{2\gamma\rho_0\omega}$$

(The calculation was carried out for a strip of this plane whose width is unity.)

In particular, for the boundary condition when $c_* \ll c_0$ we have

$$Q = A^2(\omega^2 + G^2/c_0^2)^{1/2}/(2\rho_0\omega^2)$$

2. THE NATURAL MODES OF A PLANE WAVEGUIDE WITH ABSORBING WALLS

Consider surface waves propagating in a plane waveguide formed by two rigid walls $y = h$ and $y = -h$, coated with an absorbing layer.

The boundary conditions (1.2) on the walls have the form

$$y = \pm h, \quad -\omega^2 p \pm G\partial p/\partial y = c_*^2 \partial^2 p/\partial x^2 \tag{2.1}$$

It is clear from symmetry considerations that the dependence of the natural modes on y may be symmetrical: $p \equiv \text{ch}(\gamma y)$ and antisymmetrical: $p \equiv \text{sh}(\gamma y)$. For symmetrical modes we obtain

$$k_x^2 = (G\gamma\,\text{th}\,\gamma h + \gamma^2 c_0^2)/(cc_0^2), \quad \omega^2 = (G\gamma\,\text{th}\,\gamma h + \gamma^2 c_*^2)/c \tag{2.2}$$

For antisymmetrical modes the quantity $\text{th}\gamma h$ is replaced by $\text{cth}\gamma h$. It can be seen that for an antisymmetrical mode there is a forbidden zone in the frequency band, which extends from $\omega = 0$ to $\omega = \omega^0$, and the threshold frequency is given by the equation

$$\omega^0 = [G/(ch)]^{1/2}$$

When $c_* \ll c_0$ we must neglect terms in the expressions obtained that are proportional to γ^2.

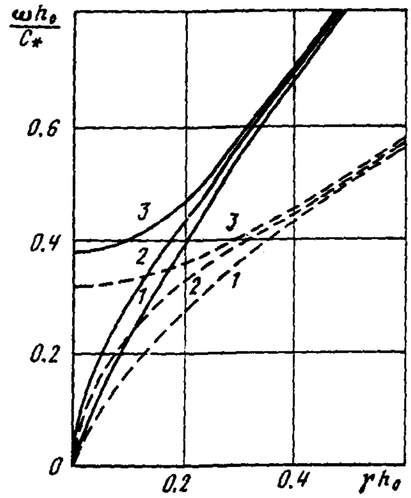

FIG. 1.

Since in the modes obtained the wave amplitude falls rapidly (exponentially) in the direction from the walls into the waveguide, it is natural to call these modes surface modes.

To illustrate the results obtained we carried out calculations for the following values of the dimensionless parameters

$$(c_*/c_0)^2 = 0.25, \quad \rho_*/\rho_0 = 0.5, \quad h/h_0 = 10.$$

In Fig. 1 we show typical curves of the dimensionless frequency $\omega h_0/c_*$ as a function of the dimensionless attenuation γh_0 for a symmetrical mode (curves 1) and an antisymmetrical mode (curves 3) in a waveguide with absorbing walls. For comparison we also show similar curves for a surface wave propagating along a single plane wall (curves 2). The dashed curves correspond to the simplified boundary conditions, while the continuous curves correspond to boundary conditions of general form.

As might have been expected, as the width of the waveguide h increases, its walls cease to affect one another and the natural modes of the waveguide become a surface wave (1.3) and (1.4).

The natural modes obtained above transfer energy along the waveguide walls. The energy flux density vector for a surface wave of amplitude A for symmetrical modes, by (2.9), is given by

$$q = (A^2 k_x (\rho_0 \omega)^{-1} \operatorname{ch}^2(\gamma y), 0)$$

For an antisymmetrical mode the quantity $\operatorname{ch}^2(\gamma y)$ must be replaced by $\operatorname{sh}^2(\gamma y)$ in the expression obtained.

The total energy flux through a strip of unit width in the $x = \text{const}$ plane is given by the equations

$$Q = \int_{-h}^{h} q_x dy = \frac{2hk_x}{\rho_0 \omega}$$

When $c_* \ll c_0$, by (2.2) we have

$$Q = 2h(1 + \gamma c_0^2 \operatorname{cth} \gamma h / G)^{1/2} (\rho_0 c_0)^{-1}$$

For antisymmetrical modes we must replace $2h$ by $(\operatorname{sh}2\gamma h - 2\gamma h)/\gamma$ and $\operatorname{cth}\gamma h$ by $\operatorname{th}\gamma h$ in the expression for the total energy flux.

In addition to natural surface modes in the waveguide with absorbing walls there can also be modes that do not attenuate with depth, similar to the natural modes of a classical waveguide. To obtain these we will seek solutions of Eq. (1.1) of the form

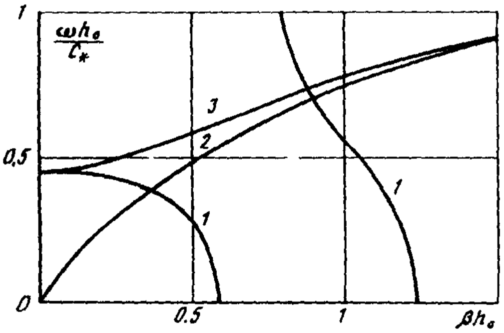

FIG. 2.

$$p = \exp[-i\omega t + ik_x x]p(y) \tag{2.3}$$

Just as for surface waves, the dependence on y can be symmetrical: $p \equiv \sin\beta y$ and antisymmetrical: $p \equiv \cos\beta y$. Then, from the boundary conditions (2.1) when $c_* \ll c_0$ we obtain

$$\omega^2 = \begin{cases} G\beta\,\mathrm{ctg}(\beta h), & \beta h \in (\pi n, \pi/2 + \pi n], \quad n = 0, 1, \ldots \\ G/h, & \beta = 0 \\ -G\beta\,\mathrm{tg}(\beta h), & \beta h \in (\pi/2 + \pi n, \pi + \pi n], \quad n = 0, 1, \ldots \end{cases} \tag{2.4}$$

Since $k_x^2 = \omega^2/c_0^2 - \beta^2$, we have $\beta < \omega/c_0$, and for each frequency ω the number of waves of the form considered is bounded by the straight line $\omega = c_0\beta$ in the (ω, β) plane. These modes have been studied in detail, and we will therefore not dwell on them here. The dispersion relations for these modes in the plane of the dimensionless parameters ($\omega h_0/c_*$, βh_0) are shown in Fig. 2 (curve 1). For comparison we also show the dispersion curves for the surface modes of waveguide with absorbing walls (curve 2 is for the symmetrical mode and curve 3 is for the antisymmetrical mode).

3. SURFACE WAVES ON THE OUTSIDE AND INSIDE OF A CYLINDRICAL WAVEGUIDE WITH ABSORBING WALLS

Consider a cylinder of radius R coated on the outside and inside with an absorbing layer and placed in an ideal liquid. By analogy with the plane case it is natural to expect that surface waves can propagate along the internal and external surface of the cylinder, i.e. the solutions attenuate rapidly with distance from the surface.

The solution of the wave equation in cylindrical coordinates

$$\frac{1}{r^2}\frac{\partial^2 p}{\partial \varphi^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial r^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0, \quad k^2 = \omega^2/c_0^2 \tag{3.1}$$

will be sought in the form

$$p = R(r)\exp(-i\omega t + ik_z z + in\varphi), \quad n = 0, \pm 1, \ldots$$

We obtain

$$\begin{gathered} x^2 R'' + xR' - (x^2 + n^2)R = 0, \quad x = \gamma r \\ R = R(x) = R(\gamma r), \quad \gamma^2 = k_z^2 - \omega^2/c_0^2 \end{gathered} \tag{3.2}$$

The boundary condition which models the absorbing layer takes the form

$$r = R_c, \quad -\omega^2 R(x) - G\partial R/\partial n = -c_*^2(n^2/r^2 + k_z^2)R(x) \tag{3.3}$$

The simplified boundary condition corresponds to (3.3) with zero right-hand side.

Waves on the external surface of the cylinder

Outside the cylinder, Eq. (3.2) has a solution possessing the required attenuation property at infinity, $R = K_n(\gamma r)$, where K_n is the modified Bessel function. From boundary condition (3.3) we obtain the dispersion law of the surface waves

$$\begin{aligned} \omega^2 &= [-G\gamma K_n'(\gamma R_c)/K_n(\gamma R_c) + c_*^2(n^2/R_c^2 + \gamma^2)]/c \\ k_z^2 &= \omega^2/c_0^2 + \gamma^2 \end{aligned} \tag{3.4}$$

In this wave energy is transferred only along the surface of the cylinder. By (1.6) for a surface wave with amplitude A the energy flux density vector and the total energy flux through a surface $z = \text{const}$ are given by the equations

$$q = \frac{A^2}{\rho_0\omega} K_n^2(\gamma r)\left(0, \frac{n}{r}, k_z\right), \quad Q = \frac{2\pi A^2 k_z}{\rho_0\omega\gamma}\int_{\gamma R_c}^{+\infty} x K_n^2(x)dx$$

Waves on the inner surface of the cylinder

Inside the cylinder Eq. (3.2) has the solution $R = I_n(\gamma r)$. The dispersion relation obtained from boundary condition (3.3) differs in this case from (3.4) by having the ratio $K_n'(\gamma R_c)/K_n(\gamma R_c)$ replaced by $I_n'(\gamma R_c)/I_n(\gamma R_c)$ while in (3.5) one must replace K_n by I_n and integrate between the limits from 0 to γR_c.

In Fig. 3 we show the results of calculations of the dispersion curves for $n = 0$ (curves 1 and 2) and $n = 1$ (curves 3 and 4) for $R_c/h_0 = 10$. Curves 1 and 3 are the dispersion relations for the external modes of the cylinder, and curves 2 and 4 are for the internal modes. Here the dashed curves correspond to the simplified boundary condition while the continuous curves correspond to boundary condition of general form.

When $R_c \to \infty$ the natural modes of a cylindrical waveguide become a surface wave (1.3)–(1.5).

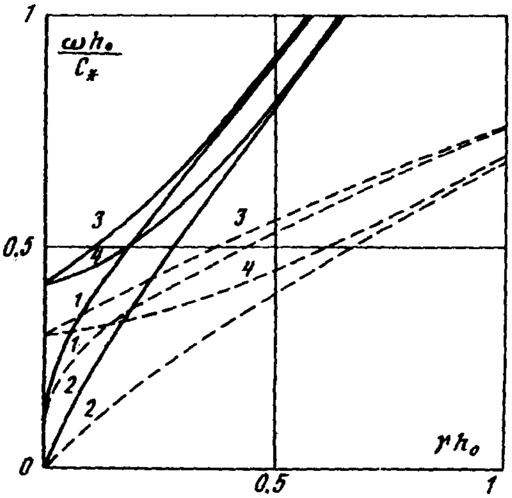

FIG. 3.

4. SURFACE WAVES IN A CYLINDRICAL LAYER

Consider the surface waves between coaxial cylinders of radii R_- and R_+ ($R_- < R_+$) coated with an absorbing layer. The simplified boundary conditions have the form

$$r = R_\pm, \quad -\omega^2 R(\gamma r) \pm G\gamma R'(\gamma r) = 0 \tag{4.1}$$

Taking the cases considered above into account, the dependence on r will be sought in the form

$$AI_n(\gamma r) + K_n(\gamma r)$$

From (4.1) we then obtain two modes of the cylindrical layer

$$\begin{aligned}
&\omega^2 = B_2 / B_1 \pm [(B_2 / B_1)^2 - B_3 / B_1]^{1/2} \\
&B_1 = a_+ - a_-, \quad B_2 = -G\gamma(b_- + b_+)/2 \\
&B_3 = (G\gamma)^2 (c_- - c_+), \quad a_\pm = I_n(\gamma R_\pm) K_n(\gamma R_\mp) \\
&b_\pm = I_n(\gamma R_\pm) K_n'(\gamma R_\mp) - I_n'(\gamma R_\pm) K_n(\gamma R_\mp) \\
&c_\pm = I_n'(\gamma R_\pm) K_n'(\gamma R_\mp)
\end{aligned} \tag{4.2}$$

It can be shown that

$$B_1 > 0, \quad B_2 > 0, \quad B_3 > 0, \quad (B_2 / B_1)^2 > B_3 / B_1$$

For R_- and R_+, which approach infinity so that $R_+ - R_- = \text{const}$, the natural modes of the cylindrical layer become natural modes of a plane waveguide.

Traditional modes, similar to the natural modes (2.3), (2.4) of a plane wave-guide, can also exist between the cylinders.

In Fig. 4 we show characteristic dispersion curves for a cylindrical layer with $R_-/h_0 = 10$, $R_+/h_0 = 15$. Curve 1 corresponds to relation (4.2) with the plus sign, curve 2 corresponds to the same relationship with a minus sign, and curve 3 corresponds to traditional modes.

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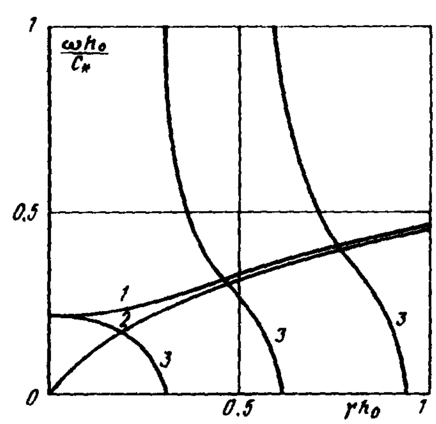

FIG. 4.

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